

THE LÉVY-KINTCHINE FORMULA FOR ROUGH PATHS

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ABSTRACT. d -dimensional Lévy processes can be considered as *rough paths* [15] with the iterated integrals understood in a suitable geometric sense; cf. Williams [23]. The expected value of the *signature* (i.e. the ensemble of all iterated integrals over a fixed time interval), which generically describes its law, is shown to admit an explicit expression. In fact, the resulting formula is precisely of Lévy–Kintchine type, with its (d -dimensional) argument omitted.

1. INTRODUCTION

Consider a d -dimensional Lévy process X with triplet (a, b, K) . The law of X at time T is an infinitely divisible measures on \mathbf{R}^d (and conversely any such measure gives rise to a Lévy process) whose Fourier transform is given by the celebrated Lévy–Kintchine formula (see e.g. [9, 5, 21, 1])

$$\mathbf{E} e^{i\langle u, X_T \rangle} = \exp \left\{ T \left(-\frac{1}{2} \langle u, au \rangle + i \langle b, u \rangle + \int (\exp(i \langle u, y \rangle) - 1 - i \langle u, y \rangle \mathbb{I}_{\{|y| \leq 1\}}) K(dy) \right) \right\}.$$

If the Lévy measure K has exponential moments, an analytic continuation is possible and the Lévy–Kintchine formula takes the form

$$(1.1) \quad \mathbf{E} e^{\langle u, X_T \rangle} = \exp \left\{ T \left(\frac{1}{2} \langle u, au \rangle + \langle b, u \rangle + \int (\exp(\langle y, u \rangle) - 1 - \langle y, u \rangle \mathbb{I}_{\{|y| \leq 1\}}) K(dy) \right) \right\}.$$

We recall that the signature [resp. expected signature] of a path [resp. process] of suitable regularity is the ensemble of all iterated integrals [resp. expectations thereof] and contains significant information of the original object, somewhat similar in spirit to Fourier series, and has gained increasing importance in pure and applied mathematics, e.g. [15, 12] and [2, 19]. For instance, in the last reference these ideas were used towards a novel numerical method, popular in financial engineering, for second order parabolic equations. Curiously enough, the following result on the expected signature of a multidimensional Lévy-processes also answers the (at first glance non-sensical) question what meaning is left to (1.1) after dropping (!) all u 's from its right-hand-side.

Theorem 1. *Assume K has moments of all orders. Then the expected signature*

$$\mathbf{E} \left[\sum_{n \geq 0} \left(\int_{0 < t_1 < \dots < t_n < T} \circ dX_{t_1} \otimes \dots \otimes \circ dX_{t_n} \right) \right] \in T((\mathbf{R}^d))$$

with values in the infinite-dimensional tensor-algebra $T((\mathbf{R}^d))$ equals precisely the right-hand-side of (1.1) upon dropping all u 's in (1.1). By this we mean that $\langle u, au \rangle$ is replaced by the (symmetric) 2-tensor a , $\langle b, u \rangle$ by the 1-tensor b and so on; all $\exp\{\dots\}$ functions are then understood via their usual power series in $T((\mathbf{R}^d))$.

A few comments are in order.

(i) The simplest way to understand the iterated integrals which appear above is in the "Wong-Zakai" sense, i.e. as the mesh-to-zero limit of the iterated integrals in which the (discontinuous) sample path X is replaced by a piecewise linear approximation based on a finite partition of $[0, T]$. This type of integration is also called of Stratonovich type (e.g. in [13] which deals with general semi-martingales) or in the sense of Marcus, [1, Ch.6], where a more intrinsic definition is given: in essence, time is stretched at each jump which is then replaced by a straight line; in the end, the size of this stretched time interval is taken to zero. This point of view was also taken in [23].

(ii) A *geometric p -rough path*, $p \in (2, 3)$, in the sense of Lyons [15] associated to such a Lévy process X was constructed in [23]; the iterated integrals can then be understood by a basic result in rough path theory (e.g. [15, First Theorem] or [11, Ch.9]. Continuity of this lifting procedure in rough path metrics will be useful in the sequel.

(iii) Theorem 1 is far from any sort of corollary of the usual Lévy-Kintchine formula. To appreciate the content of this type of result consider first $K = 0$ (case essentially contained in [8, 11]), so that X can be written in terms of a d -dimensional standard Brownian motion B , that is, $X_t = bt + \sigma B_t$ with $\sigma\sigma^T = a$. Then

$$\begin{aligned}
 (1.2) \quad \mathbf{E}X_T &= bT, \\
 \mathbf{E} \int_0^T X_t \otimes \circ dX_t &= \frac{T^2}{2} (b \otimes b) + \mathbf{E} \int_0^T \sigma B_t \otimes \circ d(\sigma B)_t \\
 &= \frac{T^2}{2} (b \otimes b) + \frac{1}{2} \langle \sigma B, \sigma B \rangle_T \\
 &= \frac{T^2}{2} (b \otimes b) + \frac{T}{2} a
 \end{aligned}$$

which is precisely consistent with

$$\begin{aligned}
 \exp \left\{ T \left(\frac{1}{2} a + b \right) \right\} &= \exp \left\{ T \left(\frac{1}{2} \sum_{i,j=1}^d a^{i,j} e_i \otimes e_j + \sum_{k=1}^d b^k e_k \right) \right\} \\
 &= \sum_{n \geq 0} \frac{T^n}{n!} \left(\frac{1}{2} \sum_{i,j=1}^d a^{i,j} e_i \otimes e_j + \sum_{k=1}^d b^k e_k \right)^{\otimes n} \in T((\mathbf{R}^d)),
 \end{aligned}$$

projected to tensor-level 1 and 2. Clearly, any direct computation of the expectation of n -fold iterated integrals is prohibitive and underlying algebraic relations between iterated integrals, in the spirit of Chen [6, 7] must be used in a crucial way.

(iv) Its own interest aside, Theorem 1 is effectively the starting point for *Cubature for Lévy-driven SDEs*, in the spirit of Lyons-Victoir [19], but now with jumps. (Needless to say, jumps are widely needed in application areas, notably in quantitative finance.) A curious feature then is that the resulting weak approximation is effectuated by (typically: piecewise smooth) paths without (!) jumps. Explicit computations of such cubature formulae are possible in many examples; this will be discussed systematically elsewhere.

(v) The afore-mentioned cubature method relies on the fact that the (truncated) expected signature arises naturally when one expands the expected value of a time- T -functional of a diffusion processes ("stochastic Taylor expansion"). This already makes it clear that the expected signature contains a significant amount of information about (the law of) the driving signal. But much more can be said: thanks to the shuffle-product [20], [17, p.50] the expected signature also provides all

moments, say of the finite-dimensional projections, of the random signature. Call μ the law of such a projection; if the moments do not grow too quickly - and in particular if the support of μ is compact - they will determine μ and then the law of the signature. (A point of view repeatedly advertized by Lyons, see e.g. [8] and also [16].) In our situation, exponential integrability of K will entail that the law of any such projection enjoys exponential integrability, and thus (see [4] and the references therein) yields a sufficient condition under which the expected signature described the law of the signature. Note that it is a different question, if (or when) the signature [resp. the law of some random signature] determines the path [resp. the law of the process]; see [12, 14] for results in this regard. But as already pointed out, as far as many applications are concerned, notably for numerics for SDEs, the expected signature contains all the relevant information.

2. EXPECTED SIGNATURE OF LEVY PROCESSES

We start with a (new) short proof of the following classical result (due to Fawcett [8], see also [19], [2, Sec.1]). Our argument emphasizes the role of IID increments of Brownian motion (and thereby naturally points to Lévy processes).

Theorem 2. *Assume $X := \sigma B$ is d -dimensional Brownian with covariance matrix $a = \sigma \sigma^T$; equivalently X is Lévy with triplet $(a, 0, 0)$. Write $S(B)_{0,T}$ for the sum of all iterated Stratonovich integrals, viewed as $T((\mathbb{R}^d))$ -valued random variable. Then*

$$\mathbf{E}S(X)_{0,T} = \exp \left(\frac{T}{2} \sum_{i,j=1}^d a^{i,j} e_i \otimes e_j \right).$$

Proof. Set $\phi_t := \mathbf{E}S(X)_{0,t}$. By Chen's relation [6] and independence of Brownian increments, also valid for the stochastic integrals built on these increments,

$$\phi_{t+s} = \phi_t \otimes \phi_s.$$

Since $\phi_t \otimes \phi_s = \phi_s \otimes \phi_t$ have $[\phi_s, \phi_t] = 0$,

$$\log \phi_{t+s} = \log \phi_t + \log \phi_s.$$

For integers m, n we have $\log \phi_m = n \log \phi_{m/n}$ and $\log \phi_m = m \log \phi_1$. It follows that

$$\log \phi_t = t \log \phi_1;$$

first $t = \frac{m}{n} \in \mathbf{Q}$, then any real t by continuity. On the other hand, for $t > 0$, Brownian scaling implies

$$\log \phi_t = \delta_{\sqrt{t}} \log \phi_1$$

where δ_λ is the dilation operator, which acts by multiplication λ^n on the n^{th} tensor level, $(\mathbf{R}^d)^{\otimes n}$. It follows that

$$\log \phi_1 \in (\mathbf{R}^d)^{\otimes 2}$$

and it remains to identify $\log \phi_1 = \pi_2(\log \phi_1)$. To this end it suffices to compute the expected signature up to level two; indeed we have already seen in the introduction, equation (1.2) with $b = 0$, that

$$\mathbf{E} \left(1 + X_{0,1} + \int_0^1 X \otimes \circ dX \right) = 1 + \frac{1}{2} \sum_{i,j=1}^d a^{i,j} e_i \otimes e_j.$$

Taking the logarithm (in the tensor algebra truncated beyond level 2) then immediately gives the desired identification. \square

Theorem 3 (Expected signature, Lévy no jumps). *Assume X is d -dimensional Lévy with triplet $(a, b, 0)$. Then*

$$\mathbf{E}S(X)_{0,T} = \exp \left\{ T \left(\frac{1}{2} \sum_{i,j=1}^d a^{i,j} e_i \otimes e_j + \sum_{k=1}^d b^k e_k \right) \right\}.$$

Proof. We offer three different arguments.

First argument: The previous proof of the drift-free case applies until the conclusion that $\log \mathbf{E}S(X)_{0,T}$ is linear in T . But the remainder of the above argument, based on Brownian scaling, fails when $b \neq 0$. We have $X_t = bt + \sigma B_t$ where $\sigma \sigma^T = a$ where B is a standard Brownian motion under some measure \mathbb{P} , say. Then

$$\mathbf{E}^{\mathbb{P}}[S(X)_{0,t}] = \exp(Ct)$$

for some $C \in T((\mathbf{R}^d))$. We will show that $\pi_k(C) = 0$ for all $k \geq 3$, the result then follows from looking explicitly at levels $k = 0, 1, 2$; see (1.2). To this end, we use

$$C = \lim_{t \rightarrow 0} \frac{\exp(Ct) - 1}{t} \implies \pi_k(C) = \lim_{t \rightarrow 0} \frac{\mathbf{E}^{\mathbb{P}}[g_t^k]}{t}$$

where $g_t^k := \pi_k(S(X)_{0,t})$ for $k \in \{0, 1, 2, \dots\}$. Now by Cameron–Martin’s theorem, $(X_t : t \in [0, T])$ is a Brownian motion under a new probability measure \mathbb{Q} . From the scaling of Brownian motion as before, we have that for all $k \geq 3$

$$\lim_{t \rightarrow 0} \frac{\mathbf{E}^{\mathbb{Q}}[|g_t^k|^2]}{t^2} = 0$$

But for any $t \in [0, T]$,

$$\mathbf{E}^{\mathbb{P}}[g_t^k] = \mathbf{E}^{\mathbb{Q}} \left[\frac{d\mathbb{P}}{d\mathbb{Q}} g_t^k \right]$$

and so by Cauchy-Schwarz inequality,

$$|\mathbf{E}^{\mathbb{P}}[g_t^k]| \leq \mathbf{E}^{\mathbb{Q}} \left[\left(\frac{d\mathbb{P}}{d\mathbb{Q}} \right)^2 \right]^{\frac{1}{2}} \mathbf{E}^{\mathbb{Q}}[|g_t^k|^2]^{\frac{1}{2}}$$

Of course, $d\mathbb{P}/d\mathbb{Q}$ (which is log-normal) has finite second moments with respect \mathbb{Q} and we easily conclude that $\pi_k(C) = 0$ for all $k \geq 3$.

Second argument, along [2]. (Sketch) Use the fact that the signature satisfies a (Stratonovich) SDE on the full tensor-algebra. Rewrite in Itô-form with Itô-Stratonovich correction, upon taking expectations, one ends up with a linear ODE on the tensor-algebra. It is explicitly solved and gives the desired result.

Third argument: Consider the deterministic ”drift” process $D : t \mapsto bt$ which is Lévy with triplet $(0, b, 0)$. It is elementary to check that

$$\mathbf{E}S(D)_{0,T} = \exp \left(T \sum_{k=1}^d b^k e_k \right).$$

Then $X_t = \sigma B_t + D_t = \sigma B_t + bt$ which is Lévy with triplet $(a, b, 0)$. The signature $Z := S(X)$ satisfies a linear Stratonovich SDE of the form

$$\begin{aligned} dZ &= Z \otimes \circ dX \\ &= V_i(Z) \circ dX^i \\ &= V_i(Z) \circ d(\sigma B)^i + W_i(Z) b^i dt \end{aligned}$$

with vector fields $V_i(z) := W_i(z) := z \otimes e_i$ (and \sum_i omitted throughout). With the usual identification of Stratonovich SDEs as RDEs driven by (Stratonovich) enhanced Brownian motion, we can now resort to splitting results for RDEs [10] to see that

$$S(X)_{0,T} = Z_{0,T} = \lim_n S(\sigma B)_{0,T/n} \otimes S(D)_{0,T/n} \otimes \dots \otimes S(\sigma B)_{T\frac{n-1}{n},T} \otimes S(D)_{T\frac{n-1}{n},T}$$

We may exchange limit and expectation (justified e.g. by a Wiener-Itô argument: after all the claim convergence is for all tensor-levels, each of which takes places in some inhomogenous Wiener-Itô chaos. As is well known, on some fixed chaos, convergence in probability is equivalent to L^q -convergence, any $q < \infty$). Then

$$\begin{aligned} \mathbf{E} S(X)_{0,T} &= \lim_n \left(\exp \left(\frac{T}{2n} \sum_{i,j=1}^d a^{i,j} e_i \otimes e_j \right) \otimes \exp \left(\frac{T}{n} \sum_{k=1}^d b^k e_k \right) \right)^{\otimes n} \\ &= \exp \left\{ T \left(\frac{1}{2} \sum_{i,j=1}^d a^{i,j} e_i \otimes e_j + \sum_{k=1}^d b^k e_k \right) \right\} \end{aligned}$$

where we used the Trotter product formula in the last step. \square

Theorem 4 (Expected signature, pure jump finite activity Lévy). *Let X be d -dimensional pure jump Lévy with triplet $(0, 0, K)$. Assume K has finite total mass $m = K(\mathbf{R}^d)$, in other words X is compound Poisson. Assume furthermore that K has moments of all orders. Then*

$$\mathbf{E} S(X)_{0,T} = \exp \left\{ T \int (\exp(y) - \mathbf{1}) K(dy) \right\}.$$

Proof. $X_t = J_1 + \dots + J_{N_t}$ is compound Poisson where (N_t) is Poisson with rate $\lambda = K(\mathbf{R}^d)$, the J_i 's are IID distributed according to K/λ . The basic remark is that conditional on n jumps over $[0, 1]$,

$$\mathbf{E} \left[S(X)_{0,1} | N_1 = n \right] = e^{J_1} \otimes \dots \otimes e^{J_n}$$

and then

$$\begin{aligned} \mathbf{E} \left[S(X)_{0,1} \right] &= \sum_{n \geq 0} e^{-\lambda} \frac{\lambda^n}{n!} \mathbf{E} (e^{J_1} \otimes \dots \otimes e^{J_n}) = \sum_{n \geq 0} e^{-\lambda} \frac{\lambda^n}{n!} (\mathbf{E} e^{J_1})^{\otimes n} \\ &= e^{-\lambda} \exp(\lambda \mathbf{E} e^{J_1}) = \exp(\lambda (\mathbf{E} e^{J_1} - \mathbf{1})). \end{aligned}$$

Here we used that scalar multiplication with $e^{-\lambda}$ is consistent with tensor multiplication of $e^{-\lambda \mathbf{1}}$, an element in $T((\mathbf{R}^d))$ which commutes with all others. Remark also that

$$\mathbf{E} e^{J_1} = \mathbf{1} + \mathbf{E} J_1 + \frac{1}{2} \mathbf{E} J_1^{\otimes 2} + \dots$$

so that the expected signature is (component-wise) well-defined provided J (equivalently: K) has moments of all orders. \square

Theorem 5. *Let X be d -dimensional Lévy with triplet (a, b, K) , K finite mass and moments of all orders (as before). Then*

$$\mathbf{E}S(X)_{0,T} = \exp \left\{ T \left(\frac{1}{2} \sum_{i,j=1}^d a^{i,j} e_i \otimes e_j + \sum_{k=1}^d b^k e_k + \int (\exp(y) - \mathbf{1}) K(dy) \right) \right\}.$$

Proof. Call Y the pure jump components, B a Brownian motion with drift so that $X = B + Y$, note that B and Y are independent. As before, the signature satisfies a linear Stratonovich type SDE of the form

$$\begin{aligned} dS(X) &= S(X) \otimes \circ dX \\ &= S(X) \otimes (\circ dB + dY), \end{aligned}$$

and it follows from splitting

$$S(X)_{0,T} = \lim_n S(B)_{0,T/n} \otimes S(Y)_{0,T/n} \otimes \dots \otimes S(B)_{T\frac{n-1}{n},T} \otimes S(Y)_{T\frac{n-1}{n},T}$$

Exchanging limit and expectation then leads to

$$\mathbf{E}S(X)_{0,T} = \lim_n \left(\exp \left\{ \frac{T}{n} \left(\frac{1}{2} \sum_{i,j=1}^d a^{i,j} e_i \otimes e_j + \sum_{k=1}^d b^k e_k \right) \right\} \otimes \exp \left\{ \frac{T}{n} \int (\exp(y) - \mathbf{1}) K(dy) \right\} \right)^{\otimes n}$$

and the result follows again by Trotter's product formula. \square

Remark 1. *A detailed analysis based on conditioning of the jumps seems difficult: condition on # jumps and jump sizes and times in $[0, T]$. Then have diffusion over $[0, T_1)$, a jump of size J_1 , diffusion over $[T_1, T_2)$ etc etc*

$$\begin{aligned} \mathbf{E} \left[S(X)_{0,T} | N_T = n, T_1, \dots, T_n; J_1, \dots, J_n \right] &= e^{CT_1} \otimes e^{J_1} \otimes e^{C(T_2-T_1)} \otimes \dots \otimes e^{J_n} \otimes e^{C(T-T_n)} \\ \mathbf{E} \left[S(X)_{0,T} | N_T = n, T_1, \dots, T_n \right] &= e^{CT_1} \otimes \mathbf{E}(e^{J_1}) \otimes e^{C(T_2-T_1)} \otimes \dots \otimes \mathbf{E}(e^{J_n}) \otimes e^{C(T-T_n)} \end{aligned}$$

Under the conditioning $N_T = 1$, the $T_1 \dots T_n$ are uniformly distributed on the simplex $\Delta_n(0, T) \equiv \{0 < t_1 < \dots < t_n < T\}$. And hence

$$\mathbf{E} \left[S(X)_{0,T} | N_T = n \right] = \int_{\{0 < t_1 < \dots < t_n < T\}} dt_1 \dots dt_n e^{CT_1} \otimes \mathbf{E}(e^{J_1}) \otimes e^{C(t_2-t_1)} \otimes \dots \otimes \mathbf{E}(e^{J_n}) \otimes e^{C(T-t_n)}.$$

Write $\mathbf{E}(e^J) := \mathbf{E}(e^{J_i})$, independent of i . By the Campell-Baker-Hausdorff formula

$$\begin{aligned} &e^{CT_1} \otimes \mathbf{E}(e^{J_1}) \otimes e^{C(t_2-t_1)} \otimes \dots \otimes \mathbf{E}(e^{J_n}) \otimes e^{C(T-t_n)} \\ &= \exp(CT + n \log \mathbf{E}(e^J) + \dots) \end{aligned}$$

where \dots contains iterated Lie brackets of $\log \mathbf{E}(e^J)$ and C . From here on it seems difficult to continue explicit computations.

Theorem 6. *The previous result remains valid when the assumption $K(\mathbf{R}) < \infty$ is replaced by the assumption that K integrates $|y|$ near 0. It also remain valid, subject to including another compensator term (" $\exp(y) - \mathbf{1} - \mathbb{I}_{\{|y| \leq 1\}} y$ ") when K integrates $|y|^2$ near 0.*

Proof. Consider a general Lévy measure, with finite moments of all order, say K , and consider the Lévy triplet (a, b, K) . Truncation of small jumps gives a family of finite Lévy measures, say $(K^\varepsilon : \varepsilon > 0)$ given by

$$K^\varepsilon(dy) := K(dy) \times \mathbb{I}_{\{|y| \geq \varepsilon\}}.$$

For fixed $\varepsilon > 0$, consider X^ε with Lévy triplet $(a, b^\varepsilon, K^\varepsilon)$ where

$$b^\varepsilon = b - \int \mathbb{I}_{\{|y| \leq 1\}} y K^\varepsilon(dy).$$

By our previous result,

$$\mathbf{E} S(X^\varepsilon)_{0,T} = \exp \left\{ T \left(\frac{1}{2} \sum_{i,j=1}^d a^{i,j} e_i \otimes e_j + \sum_{k=1}^d b^k e_k + \int (\exp(y) - \mathbf{1} - \mathbb{I}_{\{|y| \leq 1\}} y) K^\varepsilon(dy) \right) \right\}.$$

As $\varepsilon \rightarrow 0$, we have

$$\int (\exp(y) - \mathbf{1} - \mathbb{I}_{\{|y| \leq 1\}} y) K^\varepsilon(dy) \rightarrow \int (\exp(y) - \mathbf{1} - \mathbb{I}_{\{|y| \leq 1\}} y) K(dy)$$

which settles passage to the limit on the right-hand-side above. On the other hand,

$$(S(X^\varepsilon)_{0,T} : \varepsilon > 0)$$

is (for every tensor-level ...) Cauchy in probability; this is a consequence of convergence of $(X^\varepsilon, \mathbb{X}^\varepsilon)$ converges to a limiting rough path (X, \mathbb{X}) in p -variation rough path sense, any $p > 2$; cf. [23]. Note that all $\mathbb{X} = \int X \otimes \circ dX$ (similar \mathbb{X}^ε) are understood in the "geometric" sense of Marcus. This also clarifies the meaning of $S(X)$, similar $S(X^\varepsilon)$, which is understood as iterated integral in the sense of Marcus. It remains to check uniform integrability (for every tensor level) to have L^1 convergence. To this end, we show uniform integrability, in $\varepsilon \in (0, 1]$, of each component of $S(X^\varepsilon)$, i.e. of

$$\mathbf{X}_{0,T}^{\varepsilon,I} := \int_{\{0 < t_1 < \dots < t_n < T\}} \circ dX_{t_1}^{\varepsilon;i_1} \dots \circ dX_{t_n}^{\varepsilon;i_n}$$

where $I = (i_1, \dots, i_n)$ is a fixed multi-index. In fact, we shall establish uniform L^2 -integrability. To this end, set $J := I$ and note

$$|\mathbf{X}_{0,T}^{\varepsilon,I}|^2 = \mathbf{X}_{0,T}^{\varepsilon,I} \mathbf{X}_{0,T}^{\varepsilon,J} = \sum_{K \text{ shuffle of } I,J} \mathbf{X}_{0,T}^{\varepsilon,K}$$

where the sum is taken over finitely many "shuffles"; see [17, p.36]. But then

$$\begin{aligned} \mathbf{E} |\mathbf{X}_{0,T}^{\varepsilon,I}|^2 &\leq \sum_{|K| \leq 2n} \left| \mathbf{E} \mathbf{X}_{0,T}^{\varepsilon,K} \right| \\ &\leq \sum_{k=0}^{2n} \left| \pi_k \exp \left\{ T \left(\frac{1}{2} a + b + \int (\exp(y) - \mathbf{1} - \mathbb{I}_{\{|y| \leq 1\}} y) K^\varepsilon(dy) \right) \right\} \right|_{T^{(n)}(\mathbf{R}^d)} \end{aligned}$$

and it is easy to see since $K^\varepsilon(dy) := K(dy) \times \mathbb{I}_{\{|y| \leq \varepsilon\}}$ and K integrates y^2 near the origin that this estimate can be made uniform in $\varepsilon > 0$. It now follows that

$$\mathbf{E} S(X^\varepsilon)_{0,T} \rightarrow \mathbf{E} S(X)_{0,T} \text{ as } \varepsilon \downarrow 0.$$

Conclusion is, for the expect signature of a general Lévy process, subject to existence of all moments for K ,

$$ES(X)_{0,T} = \exp \left\{ T \left(\frac{1}{2} \sum_{i,j=1}^d a^{i,j} e_i \otimes e_j + \sum_{k=1}^d b^k e_k + \int (\exp(y) - \mathbf{1}_{\{|y| \leq 1\}} y) K(dy) \right) \right\}.$$

□

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